Gabor transform and intermittency in turbulence

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Intermittency effects basically limited to the viscous subrange contaminate the Gabor filtering analysis in the inertial subrange. The contamination decreases with an increase in the quality factor of the filter, and the Gabor transform coefficients of turbulent velocity have scaling properties approaching the Kolmogorov scaling. In the limit of an infinitely long inertial range, there is no anomalous scaling, i.e., no non-Gaussianity. The wavelet transform coefficients of turbulent velocity and its structure functions follow probability rules depending on the scales extracted by analyzing wavelet functions. This anomalous scaling property is produced by the intermittency effects inherent in the viscous subrange. The structure functions cannot extract properties in a pure inertial subrange.

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I. INTRODUCTION

Anselmet *et al.* [1] first observed the anomalous scaling (Paladin and Vulpiani [2] and Grassberger [3]) of the several-order moments (structure functions) of velocity differences. Their experimental results showed that the scaling exponents ζ_n for the nth-order structure functions are expressed as a nonlinear function of n. This result attracted considerable attention to the multifractal nature of turbulent velocity structures. An explanation for the nonlinearity of ζ_n was offered by Benzi *et al.* [4] and Meneveau and Sreenivasan [5] with the multifractal cascade models.

Katsuyama, Horiuchi, and Nagata [6] experimentally investigated the frequency-dependent behavior of the several-order moments of frequency-band-pass filtered signals of turbulent velocity and arrived at the following outcomes. At inertial subrange frequencies of the band-pass filter ($Q=4$), the $(2n)$th-order moments showed the anomalous scaling property, as reported by Anselmet *et al.* for the structure functions. This result means that the probability density function (PDF) of the signal, with an increase in the midband frequency, deviates gradually from the Gaussian PDF.

On the other hand, at viscous subrange (VSR) frequencies, the deviation of PDF becomes more and more remarkable with an increase in the midband frequency: at these frequencies, the band-pass signals are more intermittent than ones at the inertial subrange (ISR) frequencies. Such changes in the PDF at the inertial and viscous range frequencies reflect the viscous breakdown of the self-similar property of turbulent velocity, and indicate the intermittency basically limited in the VSR.

The Navier-Stokes equation for incompressible fluids is formally invariant under the scaling transformations, in the inviscid limit, i.e., at an infinitely large Reynolds number [7]. The scaling transformation invariance suggests that the exponents ζ_n should be expressed as a linear function of n. This linear exponent corresponds to the $-\frac{5}{3}$ power law of an energy spectrum in an infinitely long ISR (Kolmogorov's hypothesis [8]). In the present study, we answer the question of whether the nonlinear dynamics itself admits the nonlinearity of ζ_n, i.e., the anomalous scaling.

To answer the above question, we investigated the scaling property of the moments of Gabor transform coefficients for turbulent velocity signals. The Gabor transform is an analysis filter, which makes it possible to realize Q values (quality factors) much larger than that which can be attained with an electrical band-pass filter. The present study makes it clear that the nonlinearity of ζ_n observed for the structure functions and the band-pass filtered signals at small Q values is mainly due to contamination by intermittency effects inherent in the VSR. In addition, we show that at sufficiently large Q values, the PDFs of the Gabor transform coefficients of turbulent velocity signals are nearly Gaussian at the ISR frequencies.

II. GABOR TRANSFORM

The Gabor transform [9] is a short-time Fourier transform with a Gaussian window function used to localize the analyzed signals. It is expressed as

$$G[u(t)](\Omega,\sigma,t')=\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}u(t)\cos[\Omega(t-t')+\theta]\times\exp\left[-\frac{(t-t')^2}{2\sigma^2}\right]dt\ , \qquad (2.1)$$

where G denotes the Gabor transform operator, the time scale σ measures the localization of the analyzed signal, Ω is its angular frequency, and θ is its phase shift. The Gabor transform analysis extracts the Ω-frequency signals from a translating σ-size part in the time domain of the turbulent velocity sequence $u(t)$.

Each window element in a Gabor transform time-frequency domain has a time width $\Delta t=\sqrt{2}\sigma$ and an angular frequency width $\Delta\Omega=\sqrt{2}/\sigma$ [Eqs. (A3) and (A4) in the Appendix]. For each translated Gabor function, the window elements have a constant time-frequency resolution, $\Delta t \Delta\Omega = 2$; therefore, when the time scale σ is large, the Gabor transform is poor in time resolution but good in frequency resolution.

The Gabor function, given in the form of $\cos(\Omega t + \theta)\exp(-t^2/2\sigma^2)$, is an oscillatory function with the Gaussian envelope. Let M be the number of cycles of the oscillation within the interval $\Delta t = \sqrt{2}\sigma$. Then, the (angular) frequency is given as $\Omega = 2\pi M/\Delta t = \sqrt{2}\pi M/\sigma$. Since $\Delta\Omega = \sqrt{2}/\sigma$, the Q value, i.e., $\Omega/\Delta\Omega$, is defined as

$$Q = \pi M \ . \tag{2.2}$$

To actually obtain the spectrogram of a signal by filtering (e.g., using an electrical band-pass filter), one must keep its Q value constant. The same is required for the Gabor transform analysis. Now, since $Q = \Omega/\Delta\Omega$ and $\Delta\Omega = \sqrt{2}/\sigma$, the two parameters in Eq. (2.1), Ω and σ, are connected by $\Omega\sigma = \sqrt{2}Q$; therefore, when the Q value remains constant for a required frequency resolution, the Gabor function is a function of Ω (or σ) only. Then, Q is merely a parameter which specifies the quality factor of Gabor transform filtering. Hence, using $\sigma = \sqrt{2}Q/\Omega$, we can rewrite Eq. (2.1) in the form

$$G_Q u(\Omega, t') = \int_{-\infty}^{\infty} u(t) g_Q(\Omega, t - t') dt \tag{2.3}$$

with

$$g_Q(\Omega, t) = \frac{1}{\sqrt{4\pi}} \frac{\Omega}{Q} \cos(\Omega t + \theta) \exp\left[-\left[\frac{\Omega}{2Q} t\right]^2\right] , \tag{2.4}$$

where G_Q denotes the Gabor transform operator at constant Q values and $g_Q(\Omega, t)$ is a constant-Q Gabor function. The simple notation $G_Q u(\Omega, t')$ has been used instead of $G[u(t)](\Omega, \sigma, t')$ on the left-hand side of Eq. (2.1).

Since the constant-Q filtering keeps M constant, the Gabor function $g_Q(\Omega, t)$, expressing a "Gaussian-modulated wave packet," has a scaling property $g_Q(\lambda\Omega, t) = \lambda g_Q(\Omega, \lambda t)$. Hence, the constant-$Q$ Gabor transform analysis makes it possible to examine the self-similar nature of $u(t)$: the constant-Q Gabor function (2.4) is a scaling function to obtain the scaling properties of $G_Q u(\Omega, t')$. Using Eq. (2.4), we performed the Gabor transform analysis in an Ω domain for each given Q value and experimentally investigated the Q dependence of the statistical property of similar velocity structures extracted by the constant-Q Gabor transform.

The wavelet transform of $u(t)$ with respect to a given admissible mother wavelet $g(t)$ provides a wavelet domain coefficient at scale a (>0) and translation t' [10]:

$$W_g u(a, t') = \frac{1}{a} \int_{-\infty}^{\infty} u(t) g^* \left[\frac{t - t'}{a}\right] dt \ , \tag{2.5}$$

where W_g denotes the continuous wavelet transform operator and the superscript $*$ denotes complex conjugation. The wavelet transform is a broadband filtering, and breaks a signal down into scale components (i.e., scalogram). We used the following four analyzing wavelets: (i) the sombrero wavelet (the second derivative of a Gaussian function $\exp[-t^2/2]$, (ii) the first derivative of the Gaussian function, (iii) the top hat wavelet, and (iv) the double delta-function wavelet $g(t) = \delta(t + \Delta t) - \delta(t)$.

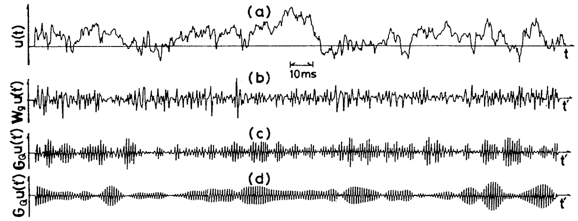

FIG. 1. The wavelet and Gabor transform coefficient of turbulent velocit signal $u(t)$. (a) The original signal $u(t)$; (b) sombrero wavelet transform coefficient; (c) Gabor transform coefficient with $Q = 4.4$; (d) with $Q = 13$, at $\Omega/2\pi = 1.0$ kHz.

The last wavelet (iv) gives a velocity difference between two points t and $t + \Delta t, u(t' - \Delta t) - u(t')$, the $(2n)$th-order moments of which are the $(2n)$th-order structure functions of $u(t)$, $S_{2n}(\Delta t) = \langle |u(t' - \Delta t) - u(t')|^{2n} \rangle$. The structure functions do not have the scaling parameter a in Eq. (2.5). However, if $S_{2n}(\Delta t)$ has a scaling property, one obtains $S_{2n}(\lambda\Delta t) = \lambda^{\zeta_{2n}} S_{2n}(\Delta t)$, λ being a positive real number. Anselmet *et al.* [1] investigated this scaling property.

The wavelet and the Gabor transform coefficients of $u(t)$ are random variables for translating mother functions. Figure 1(a) shows an original signal $u(t)$ in the time domain, 1(b) is a translation domain representation of its sombrero wavelet transform at an ISR scale, and 1(c) and 1(d) are its Gabor transform representations with different Q values at the same ISR frequency. From a comparison between Figs. 1(b) and 1(a), it follows that a sombrero wavelet transform operator removes very low frequency components and very high ones from the original signal; thus, the transformed signal in Fig. 1(b) still contains widely distributed frequency components.

As seen in Figs. 1(c) and 1(d), the Gabor transform analysis exhibits the train of wave packets, the average duration of which is long at the large Q value $(\Delta t \Delta\Omega = 2)$. This feature is a fundamental property of narrow band filtering, which exhibits oscillatory signals having frequencies specified by $\Omega/2\pi$. At significantly different Q values, the analyzed signals have no similarity. On the other hand, analyzed signals with the broadband wavelet functions cannot show amplitude-modulated oscillations. Figures 1(b)–1(d) suggest that the statistical property of the extracted signals depends on the quality factor Q in filtering. This Q dependence is investigated for a fully developed turbulent jet air flow, and leads to a correct understanding of intermittency.

III. Q DEPENDENCE OF GABOR TRANSFORM COEFFICIENTS

The streamwise component of turbulent velocity in the jet flow, $u(t)$, was detected with a hot-wire anemometer and the detected signal was digitized with 12-bit resolution. The time sequence of $u(t)$ was sampled at seven different intervals of 2, 5, 10, 30, 100, 300, and 1000 μs. The wavelet and the Gabor transform coefficients were numerically calculated from the records of 1.6×10^6 data obtained at each sampling rate. The mean flow velocity at the measuring position was $U=15$m/s, and the turbulence Reynolds number $R_\lambda=270$. Figure 2 shows the experimentally obtained one-dimensional energy-frequency spectral distribution of $u(t)$, $E_1(f)$, where f is a frequency. In the ISR ranging from f_0 to $63f_0$, $E_1(f)$ approximately shows the $-\frac{5}{3}$ power law.

The $(2n)$th-order moments of the Gabor transform coefficients $G_Q u(\Omega,t')$, $M_{2n}(\Omega,Q)$, are given as

$$M_{2n}(\Omega,Q)=\frac{1}{2T}\int_{-T}^{T}[G_Q u(\Omega,t')]^{2n}dt' , \tag{3.1}$$

where $2T$ is the total time in the translation and $n=1,2,\ldots$. The moments $M_{2n}(\Omega,Q)$ were evaluated across an Ω domain at each of the given Q values. The moments are independent of the phase shift θ on the right-hand side (rhs) of Eq. (2.4). In the experiment, we chose $\theta=0$.

Figures 3(a) and 3(b) show the log-log plots of $M_{2n}(\Omega,Q)$ versus $\Omega/2\pi(=f)$ at the different Q values, where the arrows indicate the minimal and maximal frequencies of the ISR of $E_1(f)$. The plots in 3(a) and 3(b) have the same scaling range from f_0 to $63f_0$. The scaling range corresponds to the ISR of $E_1(f)$ (Fig. 2), irrespective of the Q value. The scaling property disappears in the high frequency range such that $f>63f_0$. The straight lines in Fig. 3 were obtained by applying the least squares method to the data in the ISR. The moments $M_{2n}(\Omega,Q)$ have a scaling property expressed by

$$M_{2n}(\Omega,Q)\propto\Omega^{-\zeta_{2n}(Q)} , \tag{3.2}$$

where $\zeta_{2n}(Q)$ is a scaling exponent.

The scaling property of $M_{2n}(\Omega,Q)$ was investigated for Q values ranging from 1.6 to 89. The scaling exponents $\zeta_{2n}(Q)$ depended on the Q value. Figure 4 shows the plots of $\zeta_{2n}(Q)$ versus $2n$. The $\zeta_{2n}(Q)$ curves are nonlinear at all the given Q values. As the Q value gets larger, the nonlinear curves of $\zeta_{2n}(Q)$ have a tendency to shift toward the straight line $\zeta_{2n}=2n/3$, the Kolmogorov scaling. This behavior is attributed to the effect that the narrower frequency bands decrease contamination by the intermittency effects in the VSR: at small Q values, the scaling property in the ISR includes contamination by the intermittency effects.

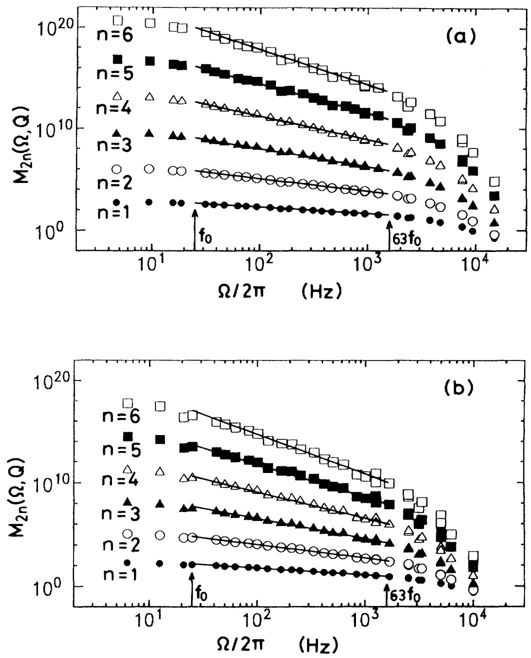

FIG. 3. Log-log plots of $M_{2n}(\Omega,Q)$ vs $\Omega/2\pi$ at $Q=27$ (a) and 89 (b). Arbitrary units on the ordinate.

Figure 5 shows the Q dependence of $\zeta_{2n}(Q)$ for $n=1,2,\ldots,6$. The exponents $\zeta_{2n}(Q)$, with an increase in the Q value, have a tendency to approach the constant values $2n/3$. Concerning the second moment $M_2(\Omega,Q)$, it is easy to theoretically show that its scaling exponent $\zeta_2(Q)$ should depend on the Q value. As shown in the Appendix, $M_2(\Omega,Q)$ is given by Eq. (A9). Here, we introduce a function with a unity integral, $\delta(\omega-\Omega,\sigma)$:

$$\delta(\omega-\Omega,\sigma)=\frac{\sigma}{\sqrt{\pi}}\exp[-\sigma^2(\omega-\Omega)^2] . \tag{3.3}$$

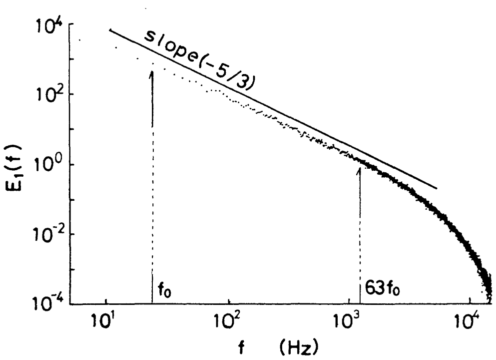

FIG. 2. One-dimensional frequency-energy spectrum $E_1(f)$. The arrows indicate both ends of its inertial range. Arbitrary units on the ordinate.

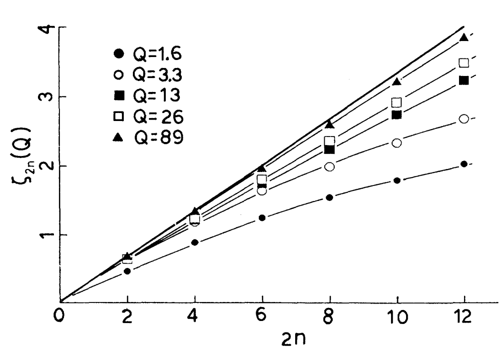

FIG. 4. Plots of $\zeta_{2n}(Q)$ vs $2n$ for different Q values, where the straight line is $\zeta_{2n}=2n/3$.

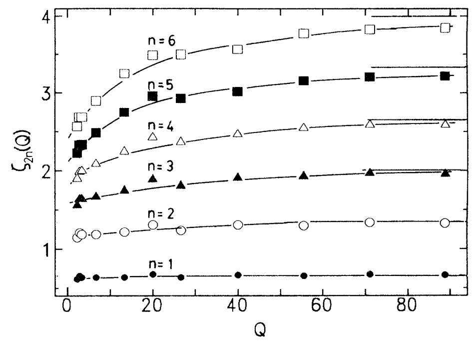

FIG. 5. The Q dependence of $\zeta_{2n}(Q)$.

This function $\delta(\omega-\Omega,\sigma)$, with increasing σ, differs appreciably from zero only over a smaller and smaller ω interval around $\omega=\Omega$. In the limit $\sigma\rightarrow\infty$, $\delta(\omega-\Omega,\sigma)$ is the Dirac delta function.

By using Eq. (3.3) and $\sigma=\sqrt{2}Q/\Omega$, we rewrite Eq. (A9) in the form

$$M_2(\Omega,Q)\propto\frac{1}{\sqrt{2}Q}\int_{-\infty}^{\infty}E_1(\omega)\delta\left[\frac{\omega}{\Omega}-1,\sqrt{2}Q\right]d\omega\ . \tag{3.4}$$

This expression shows that the scaling exponent of $M_2(\Omega,Q)$ should depend on the Q value. However, as demonstrated in Fig. 5, the Q dependence becomes more and more conspicuous at the order numbers larger than 2. Since $E_1(\omega)\propto|\omega|^{-5h}$ in the limit $R_\lambda\rightarrow\infty$, the limit $Q\rightarrow\infty$ leads Eq. (3.4) to

$$M_2(\Omega,Q)\propto\Omega^{1-5h}\ . \tag{3.5}$$

Then, the second moment $M_2(\Omega,Q)$ exactly obeys the scaling law.

The non-Gaussianity of the PDF can be described by the following factor:

$$\gamma_{2n}(Q)=\frac{(2n-1)!!}{2}\frac{[M_2(\Omega,Q)]^n}{M_{2n}(\Omega,Q)}\ , \tag{3.6}$$

where $(2n-1)!!=(2n-1)(2n-3)\times\cdots\times3\times1$. When the PDF of $G_Qu(\Omega,t')$ is exactly Gaussian, one has $\gamma_{2n}(Q)=\frac{1}{2}$ at every order. The intermittency of turbulent velocity causes the PDF to deviate from the Gaussian PDF. In this sense, we called this factor $\gamma_{2n}(Q)$ a $(2n)$th-order "intermittency factor" [6].

Figures 6(a) and 6(b) show the log-log plots of $\gamma_{2n}(Q)$ versus $\Omega/2\pi$ at the two different Q values. The frequency range in which $\gamma_{2n}(Q)$ takes the fixed value of 0.5 for all the given orders is quite narrow at the small Q value [6(a)], but at the large Q value [6(b)], it extends to the high frequency end $63f_0$ of the ISR of $E_1(f)$. At these frequencies at which $\gamma_{2n}(Q)\cong0.5$ at all the given n values, the PDFs of $G_Qu(\Omega,t')$ remain nearly Gaussian. At $Q=89$ [6(b)], the deviation of the PDF from the Gaussian PDF begins to occur at $f>63f_0$: the PDFs are nearly Gaussian in the ISR ranging from f_0 to $63f_0$. On the other hand, at $Q=6.7$ [6(a)], the deviation begins to occur at the low frequency located in the ISR and is mainly due to contamination by the intermittency effects.

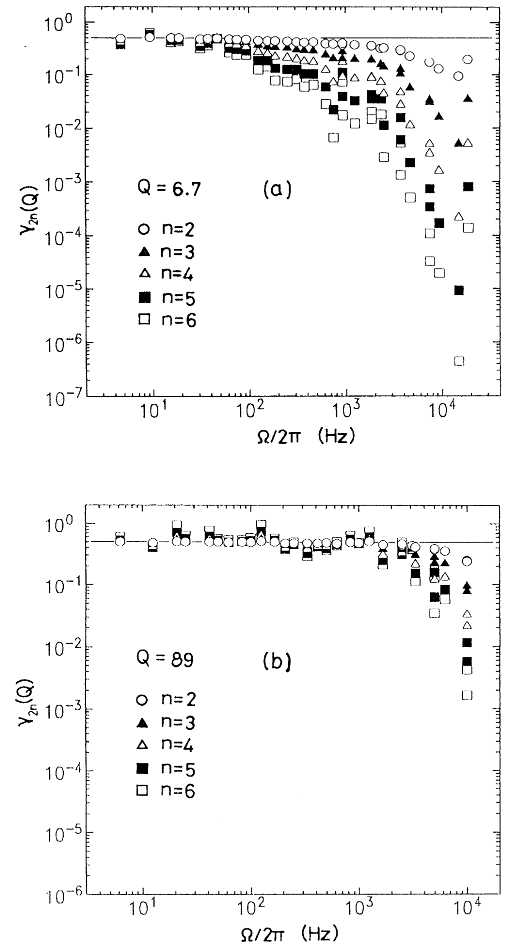

FIG. 6. Log-log plots of $\gamma_{2n}(\Omega)$ vs $\Omega/2\pi$ at two Q values: (a) $Q=6.7$; (b) $Q=89$.

Let $M_{2n}(a)$ be the $(2n)$th order moments of $\mathbf{W}_gu(a,t')$. If we substitute a length scale l, given as $l=2aU$ with the mean flow velocity U at the measuring point, for the time scale a, then $lf=U$ holds approximately. At the length scales l corresponding to the ISR frequencies, $M_{2n}(l)$ approximately satisfies the scaling property:

$$M_{2n}(l)\propto l^{\zeta_{2n}}\ , \tag{3.7}$$

where ζ_{2n} is the scaling exponent of $M_{2n}(l)$. The structure function wavelet (iv) also presented the same scaling behavior.

Figure 7 shows the plots of ζ_{2n} as a function of $2n$. For all the given wavelet functions, ζ_{2n} takes the same behavior. It is, therefore, possible to say within our examined range of $2n$ that the deviation of the PDF from the Gaussian PDF depends very little on the type of analyzing wavelet. The ζ_{2n} curves are similar to the $\zeta_{2n}(Q)$ curve at $Q=3.3$ (Fig. 4). The structure functions,

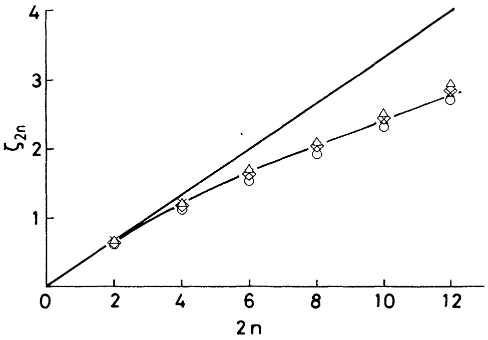

FIG. 7. Plots of ζ_{2n} vs $2n$. (◇) Mexican hat: (○) the first derivative of Gaussian function; (×) top hat; (△) structure function.

therefore, include contamination by the intermittency effects inherent in the VSR.

IV. CONCLUSIONS

In the limit of an infinitely long ISR, there appears to be no anomalous scaling. Then, the linear scaling exponent $\zeta_{2n}(Q)=2n/3$, the Kolmogorov scaling, is expected. Real turbulence exhibits a finite width of the ISR, at higher frequencies in which the viscous effects, the energy dissipation and the work by viscous shear stress, cannot be neglected and become more and more significant; therefore, the VSR cannot be clearly isolated from the ISR, and thus the ISR has a high-frequency-sided region which contains properties of both a pure ISR and a VSR. At moderate Reynolds numbers, the non-Gaussianity and the anomalous scaling are due to potential contamination by the VSR of the linear scaling expected in a pure ISR limit.

If the PDF of $G_Q u(\Omega,t')$ were exactly Gaussian at every ISR frequency, one would obtain $\zeta_{2n}(Q)=2n/3$ for all n. In such a hypothetical situation, the behavior of $G_Q u(\Omega,t')$, in the statistical sense, must be completely self-similar in the pure ISR. However, since the fluid is viscous, real turbulence at finite R_λ values cannot take completely self-similar velocity structures [11]. Hence, in the Gabor transform analysis with finite frequency bands, the viscous effects make $\zeta_{2n}(Q)$ nonlinear. On the other hand, at higher frequencies in the VSR, the behavior of $G_Q(\Omega,t')$ becomes intrinsically intermittent irrespective of R_λ. The fast falloff behavior in the VSR of $E_1(f)$ is evidence of the appearance of the intermittency effects.

The scaling property is an attribute of nonlinear effects on turbulent fluid motion. Real turbulence includes effects due to the viscous breaking of the scaling transformation invariance in the inviscid limit of the Navier-Stokes equation. At finite R_λ values, the scaling law holds approximately for the Gabor and wavelet analysis, and not exactly by any means. A decrease in R_λ worsens the approximation, and the scaling law holds exactly only in the limit of an infinitely long ISR. Further investigations into the R_λ dependence of the scaling property are needed to make sure of the validity of the above inference (e.g., Grossmann *et al.* [12]).

APPENDIX

The Gabor function $g(t)$ is expressed in terms of Ω and σ like in Eq. (2.1) as

$$g(t-t')=\frac{1}{\sqrt{2\pi}\sigma}\frac{1}{2}[e^{i\Omega(t-t')+i\theta}+\text{c.c.}]\times\exp\left[-\frac{(t-t')^2}{2\sigma^2}\right], \quad (A1)$$

where the term c.c. denotes complex conjugation. The Fourier transform of $g(t-t')$, $G(\omega)=\int_{-\infty}^{\infty}\exp(-i\omega t)g(t-t')dt$, is given as

$$G(\omega)=\frac{1}{2}e^{-i\omega t'}\left\{e^{i\theta}\exp\left[-\frac{\sigma^2(\omega-\Omega)^2}{2}\right]+e^{-i\theta}\exp\left[-\frac{\sigma^2(\omega+\Omega)^2}{2}\right]\right\}. \quad (A2)$$

If $t-t'=\Delta t$ in Eq. (A1) and $\Delta t/\sqrt{2}\sigma=1$, one has

$$\Delta t=\sqrt{2}\sigma . \quad (A3)$$

Next, if $\omega-\Omega=\Delta\Omega$ in Eq. (A2) and $\Delta\Omega\sigma/\sqrt{2}=1$, one gets

$$\Delta\Omega=\frac{\sqrt{2}}{\sigma} . \quad (A4)$$

Since $g(t-t')=(1/2\pi)\int_{-\infty}^{\infty}G(\omega)\exp(i\omega t)d\omega$, the Gabor transform coefficient, $G_Q u(\Omega,t')=\int_{-\infty}^{\infty}u(t)g(t-t')dt$, is expressed as

$$G_Q u(\Omega,t')=\frac{1}{4\pi}\int_{-\infty}^{\infty}e^{-i\omega t'}U^*(\omega)\left\{e^{i\theta}\exp\left[-\frac{\sigma^2(\omega-\Omega)^2}{2}\right]+e^{-i\theta}\exp\left[-\frac{\sigma^2(\omega+\Omega)^2}{2}\right]\right\}d\omega . \quad (A5)$$

The second moment of $G_Q u(\Omega,t')$ is given as

$$M_2(Q,\Omega)=\frac{1}{2T}\int_{-T}^{T}|G_Q u(\Omega,t')|^2dt' . \quad (A6)$$

Substituting Eq. (A5) into Eq. (A6) and then carrying out the integral with respect to t', we get

$$M_2(Q,\Omega)=\left[\frac{1}{4\pi}\right]^2\int\int_{-\infty}^{\infty}\frac{\sin(\omega-\omega')T}{(\omega-\omega')T}U^*(\omega)U(\omega')\left\{e^{i\theta}\exp\left[-\frac{\sigma^2(\omega-\Omega)^2}{2}\right]+e^{-i\theta}\exp\left[-\frac{\sigma^2(\omega+\Omega)^2}{2}\right]\right\}$$
$$\times\left\{e^{-i\theta}\exp\left[-\frac{\sigma^2(\omega'-\Omega)^2}{2}\right]+e^{i\theta}\exp\left[-\frac{\sigma^2(\omega'+\Omega)^2}{2}\right]\right\}d\omega\,d\omega' . \quad \text{(A7)}$$

From the property of the sin*c* function with a sufficiently large value of T, we obtain

$$M_2(Q,\Omega)=\left[\frac{1}{4\pi}\right]^2\int_{-\infty}^{\infty}|U(\omega)|^2\left\{\exp[-\sigma^2(\omega-\Omega)^2]+\exp[-\sigma^2(\omega+\Omega)^2]\right.$$
$$\left.+2\cos(2\theta)\exp\left[-\frac{\sigma^2(\omega-\Omega)^2}{2}\right]\exp\left[-\frac{\sigma^2(\omega+\Omega)^2}{2}\right]\right\}d\omega . \quad \text{(A8)}$$

The vanishing integral of $g(t)$ is that $G(\omega)$ must take a zero value at $\omega=0$; therefore, Eq. (A2) must satisfy $\exp(-\sigma^2\Omega^2/2)\simeq 0$. Here, taking $\sigma=\sqrt{2}Q/\Omega$ into account, one finds a condition such that $\exp(-Q^2)\simeq 0$, which technically requires that $Q>2.0$. Under this condition, the third term on the rhs of Eq. (A8) can be neglected. In addition, the first and second terms make the same contribution to $M_2(Q,\Omega)$, and $|U(\omega)|^2=E_1(\omega)$. Thereby, Eq. (A8) can be reduced to

$$M_2(Q,\Omega)=\left[\frac{\sqrt{2}}{4\pi}\right]^2\int_{-\infty}^{\infty}E_1(\omega)\exp[-\sigma^2(\omega-\Omega)^2]d\omega . \quad \text{(A9)}$$

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